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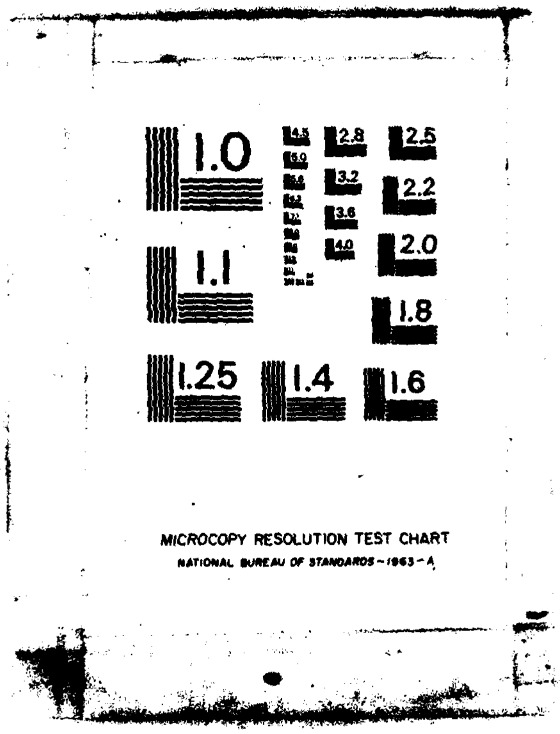
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NOISE

Abstract

This report explains the term "noise" from the point of view of probability and statistics. Namely, noise is regarded as a random process. Two special cases of noise, Gaussian noise and white noise, are discussed.

Key words: Non-linear filtering, Gaussian noise, power spectral density, white noise

The term "noise" was first used in communications engineering as a result of the undesired acoustic effects accompanying spontaneous electric fluctuations in receivers. Ever since the advent of electric communications, communications engineers have searched to reduce the electric noise in communications systems. Nowadays the idea of noise has been widely used in many other fields where no acoustic effect is involved. Roughly speaking, noise means something which interferes with the desired signal. There is a variety of sources of noise. For example, the fundamental source of electric noise is the quantized electric charge. (Its graininess causes the voltage fluctuations in electric circuits.) Another common example is the computer generated noise due to roundoff error. In many statistical methods such as regression analysis and categorical data analysis, the noise is the randomly fluctuating part (e.g. due to sampling) and the signal is the unknown deterministic parameters. To understand the noise structure and to extract the signal from the noisy environment are the most important objectives in many studies.

Noise is usually treated as a stochastic process of irregular fluctuations. In applications, the noise process is often assumed to be stationary and ergodic*. That is, its statistical properties can be completely characterized by just one sample over a long period. Many tools used in time series analysis* [4, Part II] are very useful in

analyzing noise. For example, fast Fourier transform* (FFT) [10, Chapter 6] is a powerful computational tool in estimating noise power spectra. The analysis of autocorrelation and partial autocorrelation functions* is useful in fitting autoregressive-moving average* (ARMA) models to the noise process.

GAUSSIAN NOISE

In many cases, noise is best described as a Gaussian process* (e.g. thermal noise and shot noise in electronic systems). That is, the joint distribution of noise random variables at any set of time points is multivariate normal. In other words, the noise process can be completely characterized by its autocorrelation function, or equivalently its spectral distribution if it is stationary. The assumed normality can be justified by the central limit theorem* if the noise is composed of many small independent (or weakly dependent) random effects. The main advantage of using Gaussian assumption is that the best linear estimator of the signal is optimal under the criterion of mean squared error. (Of course, the signal needs to be Gaussian, too). That is to say, there is no need to consider non-linear theory in signal estimation in Gaussian systems. In signal detection, the likelihood ratio statistic is an optimal test statistic, and many results have been established on absolute continuity and the Radon-Nikodym derivative between the two measures

induced by pure noise and signal plus noise, when both measures are Gaussian [7, Chapter 3;13]. For example, Feldman and Hajek derived the dichotomy theorem that two Gaussian measures are either singular or equivalent. Rice [11,12] studied the behavior of sample paths of stationary Gaussian noise with zero mean in continuous time. Suppose that $X(t)$ is stationary Gaussian with $EX(t) = 0$ and $EX(t)X(t+s) = R(s)$. The expected number of zeros per second in sample paths is $2[\int_0^\infty f^2 w(f)df / \int_0^\infty w(f)df]^{1/2}$ where $w(f) = 4\int_0^\infty R(t)\cos 2\pi ft dt$ is the power spectral density and f is frequency in cycles per second. However, there is not much known about the distribution of the distance between two successive zeros. The expected number of local maxima per second in sample paths is $[\int_0^\infty f^4 w(f)df / \int_0^\infty f^2 w(f)df]^{1/2}$. When the noise $X(t)$ is narrow-band (i.e. the spectrum $w(f)$ vanishes except in a small region), the envelope of $X(t)$ has the Rayleigh distribution which is defined to be the distribution of the square root of a random variable with the chi-squared distribution of degree 2.

It is worth noting that even though optimal statistics can be obtained under the Gaussian condition, robust statistics* are desired so that decisions based on the statistics are less sensitive to the Gaussian assumption. Various kinds of non-Gaussian noise take place in different situations. Some are generated from Gaussian noise through non-linear devices. For example, the output voltage of

noise has the Rayleigh distribution when narrow-band Gaussian thermal noise is applied to an evelope detector. Another example is quantization noise which has the uniform distribution from -d/2 to d/2 where d is a unit quantization step. Quantization noise occurs when analog signals are converted to digital form. One more example is impulse noise. Impulse noise is a generalized random process composed of short bursts which occur at random time points with random amplitudes. In the analysis of electroencephalographic (EEG) wave recordings of brain activity, the EEG recordings are a Gaussian signal process of brain activity plus a Poisson process of impulse noise due to muscle contractions by nerve impulses [8].

WHITE NOISE

White noise is a stationary stochastic process with constant spectral density. The term "white" is borrowed from optics where "white light" has been used to signify uniform energy distribution among the colors. (Actually the analogy is not correct since in optics the uniform energy distribution of white light is based on wavelength (the reciprocal of frequency) rather than frequency [3, p. 14].) Discrete-time white noise is simply an uncorrelated wide-sense stationary time series with zero mean. Autoregressive-moving average* (ARMA) processes are driven by discrete-time white noise. Continuous-time white noise

X(t) satisfies formally $EX(t) = 0$ and $EX(t)X(s) = \sigma^2\delta(t-s)$ where $\delta(\cdot)$ is the Dirac delta function*. In the following, we only consider continuous-time white noise. Since entirely flat spectral density distribution implies infinite power, white noise does not exist in practice. Nevertheless, the use of a white noise model is justified in many aspects. Many real data sets, such as aircraft flight test data, radar return data and passive sonar detection data, involve an additive random noise which has large bandwidth compared to that of the signal. In some other problems, noise may be best described as a linear transformation of white noise. More importantly, it is often much easier analytically and computationally to deal with white noise (e.g. Kalman filter*). A crucial problem of robustness arises: Can the results based on a white noise model effectively approximate those based on a real case where the noise involved has large but finite bandwidth? When only linear operations on the data are considered, the answer is yes, i.e. the results based on a white noise model are consistent with the asymptotic case where the noise bandwidth tends to infinity in any way desired. However, serious difficulties arise in interpretation when non-linear operations on the data have to be considered [2]. The problem of robustness has yet to be further investigated in the non-linear case.

Let us take a close look at the following non-linear filtering problem. Let $Z(t)$ and $X(t)$ $(t \geq 0)$ be the mutually

independent signal and white noise, respectively. Let $Y(t) = Z(t) + X(t)$ be the observation process. The classic filtering problem is to calculate the conditional expectation (or distribution) of $Z(t)$ given $\{Y(s): o \leq s \leq t\}$. Traditionally, white noise $X(t)$ is treated as the formal derivative of the Wiener process* $W(t) = \int_0^t X(s)ds$. The Wiener process is well defined in terms of the Wiener measure on $C[0,T]$ (= the set of all continuous real-valued functions on $[0,T]$) where T is the time span. Now, the filtering problem can be solved using Ito integral* which allows non-linear operations. The conditional expectation of $Z(t)$ given $\{Y(s): o \leq s \leq t\}$ is uniquely determined up to a null subset of sample paths. Unfortunately, the Wiener process sample paths are of bounded variation with probability zero and all the physical sample paths are of bounded variation. Thus non-linear filtering theory cannot be applied in practice unless we are able to choose a particular version of the conditional expectation which is defined everywhere (not just almost everywhere) and continuous in sample path with respect to the supremum norm on $C[0,T]$. This is another robustness problem. It has been taken up by several authors [5,6,9]. Balakrishnan [2] took a different functional approach which avoids the above problem. He developed non-linear white noise theory in which white noise is defined on $L_2[0,T]$ (= the set of all square integrable real-valued functions on $[0,T]$). Specifically, let C be the class of cylinder sets in $L_2[0,T]$ with

Borel bases in finite dimensional subspaces. A weak distribution is a finitely additive probability measure on C which is countably additive on any class of cylinder sets with bases in the same finite dimensional subspace. Now, white noise is defined to be the triple $(L_2[0,T], C, \mu_G)$ where μ_G is the weak distribution (Gaussian measure) defined by the characteristic function*

$$\int_{L_2[0,T]} \exp[i\int_0^T h(t)x(t)dt]d\mu_G(x) = \exp[-\frac{1}{2}\int_0^T h^2(t)dt]$$

for all $h \epsilon L_2[0,T]$. Based on this construction of white noise, some results on likelihood ratio, innovation process and conditional density are derived [1, Chapter 6; 2].

SIGNAL PROCESSING IN THE PRESENCE OF NOISE

The effect of noise on information transmission is the main subject of statistical communication theory*. Information theory*, signal detection and signal estimation (filtering and smoothing) are three important topics. See the article on STATISTICAL COMMUNICATION THEORY for a detailed exposition. Spline approximation* is another subject related to signal estimation in the presence of noise. It is a method of recovering a smooth (signal) function f(t) $(o \leq t \leq T)$ when only discrete, noisy measurements $y_i = f(t_i) + \varepsilon(t_i)$ $(i = 1,2,\ldots,n)$ are available. What is known about f(t) is that it is smooth, e.g. $f''(t) \in L_2[0,T]$.

The spline estimator of f is the spline function which minimizes $\frac{1}{n}\sum_{i=1}^{n}(f(t_i) - y_i)^2 + \lambda\int_0^T (f''(s))^2 ds$ where the smoothing parameter λ is determined by the generalized cross-validation method [14]. This estimator may be regarded as a low-pass smoother in a general sense. The estimator has the remarkable property that the higher order derivatives of it are good estimators of those of f.

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This report explains the term "noise" from the point of view of probability and statistics. Namely, noise is regarded as a random process. Two special cases of noise, Gaussian noise and white noise, are discussed.